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Interaction with the Absorber as the Mechanism of Radiation†*

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"We must, therefore, be prepared to find that further advance into this region will require a still more extensive renunciation of features which we are accustomed to demand of the space time mode of description."—Niels Bohr[1]

PAST FAILURE OF ACTION AT A DISTANCE TO ACCOUNT FOR THE MECHANISM OF RADIATION

It was the 19th of March in 1845 when Gauss described the conception of an action at a distance propagated with a finite velocity, the natural generalization to electrodynamics of the view of force so fruitfully applied by Newton and his followers. In the century between then and now what obstacle has discouraged the general use of this conception in the study of nature?

The difficulty has not been that of giving to the idea of propagated action at a distance a suitable embodiment of electromagnetic equations. This problem, to be true, remained unsolved to Gauss and his successors for three quarters of the century. But the formulation then developed by Schwarzschild and Fokker, described and amplified in another article,[2] demonstrated that the conception of Gauss is at the same time mathematically self consistent, in agreement with experience on static and current electricity, and in complete harmony with Maxwell's equations.

To find the real obstacle to acceptance of the tool of Newton and Gauss for the analysis of forces, we have to go beyond the bounds of steady-state electromagnetism to the phenomena of emission and propagation of energy. No branch of science has done more than radiation physics to favor the evolution of present concepts of field or more to pose difficulties for the idea of action at a distance. The difficulties have been twofold—to obtain a satisfactory account of the field generated by an accelerated charge at a

* A preliminary account of the considerations which appear in this paper was presented by us at the Cambridge meeting of the American Physical Society, February 21, 1941, Phys. Rev. **59**, 683 (1941).

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† *Introductory Note.*—In commemoration of the sixtieth birthday of Niels Bohr it had been hoped to present a critique of classical field theory which has been in preparation since before the war by the writer and his former student, R. P. Feynman. The accompanying joint article, representing the third part of the survey, is however the only section now finished. The war has postponed completion of the other parts. As reference to them is made in the present section, it may be useful to outline the plan of the survey.

The motive of the analysis is to clear the present quantum theory of interacting particles of those of its difficulties which have a purely classical origin. The method of approach is to define as closely as one can within the bounds of classical theory the proper use of the field concept in the description of nature. Division I is intended first to recall the possibility of idealizing to the case of arbitrarily small quantum effects, a possibility which is offered by the freedom of choice in the present quantum theory for the dimensionless ratio (quantum of angular momentum) (velocity of light)/(electronic charge)^2; then however to recognize the possible limitations placed on this analysis by the relatively large value, 137, of the ratio in question in nature; and finally to present a general summary of the conclusions drawn from the more technical parts of the survey. The plan of the second article is a derivation and resumé of the theory of action at a distance of Schwarzschild and Fokker, to prepare this theory as a tool to analyze the field concept. From the correlation of the two points of view, one comes to Frenkel's solution of the problem of self-energy in the classical field theory and to new expressions for the energy of electromagnetic interaction in the theory of action at a distance. The third division, which is published herewith, is an analysis of the mechanism of radiation believed to complete the last tie between action at a distance and field theory and to remove the obstacle which has so far prevented the use of both points of view as complementary tools in the description of nature. It is the plan of a subsequent division to discuss the problems which arise when the fields are regarded as subordinate entities with no degrees of freedom of their own. An infinite number of degrees of freedom are found to be attributed to the particles themselves by the theory of propagated action at a distance. However, it appears that the additional modes of motion are divergent and have on this account to be excluded by a general principle of selection. Acceptance of this principle leads to the conclusion that the union of action at a distance and field theory constitutes the natural and self-consistent generalization of Newtonian mechanics to the four-dimensional space of Lorentz and Einstein.—J. A. W.

[1] Niels Bohr, *Atomic Theory and the Description of Nature* (Cambridge University Press, Teddington, England, 1934).

[2] Unpublished, see Introductory Note.

remote point and to understand the source of the force experienced by the charge itself as a result of its motion:

(a) An accelerated charge generates a field given, according to the formulation of Schwarzschild and Fokker, by half the usual retarded solution of Maxwell's equations, plus half the advanced solution. From the presence of the advanced field in the expression for the electric vector, it follows that a distant test body will experience a premonitory force well before the source itself has commenced to move. To avoid a conclusion so opposed to experience Ritz[3] and Tetrode[4] proposed to abandon the symmetry in time of the elementary law of force. However, it was then necessary to give up the possibility to derive the equations of motion and all the electromagnetic forces consistently from a single unified principle of least action like that of Fokker. More important, the sacrifice made to alleviate one difficulty of the theory of action at a distance did not help to solve the other, the problem of the origin of the force of radiative reaction.

(b) Experience indicates that an accelerated charge suffers a force of damping which is simultaneous with the moment of acceleration. However, the theory of action at a distance predicts that an accelerated charge in otherwise charge-free space will experience no electric force. To exclude the acceleration and thus to avoid the issue does not appear reasonable. Uncharged particles can be present and can accelerate the charge via gravitational forces. It seems just as difficult to explain the reactive force when other charged particles are present. They will indeed be set into motion and will act back on the source. However, if these elementary interactions have the purely retarded character assumed by Ritz, and also by Frenkel,[5] the reaction will arrive at the accelerated particle too late and will have the wrong magnitude[6] to produce the damping phenomenon. On the other hand, interactions symmetrical between past and future—the half-retarded, half-advanced fields of the unified theory of action at a distance—have so far appeared to be equally incapable of accounting for the observed force of radiative reaction, with its definitely irreversible character.

It is clear why the viewpoint of Newton and Gauss has not been generally applied in recent times; it has so far failed to give a satisfactory account of the mechanism of radiation.

The failure of action at a distance cannot pass unnoticed by field theory. The two points of view, according to the thesis of the present critique, are not independent, but mutually complementary. Consequently field theory, too, faces in the radiation problem a significant issue: does this theory give an explanation for the observed force of radiative reaction which can be translated into the particle mechanics of Schwarzschild and Fokker, or does it likewise fail to provide a complete picture of the mechanism of radiation?

In attacking the radiation problem our first move, following the above reasoning, is to review the status of the reaction force in existing classical field theory. No more intelligible clue is found to the physical origin of the force in this theory than in the theory of action at a distance. Stopped on this approach, we take up a suggestion made long ago by Tetrode that the act of radiation should have some connection with the presence of an absorber. We develop this idea into the thesis that the force of radiative reaction arises from the action on the source owing to the half-advanced fields of the particles of the absorber; or, more briefly, that radiation is a matter as much of statistical mechanics as of pure electrodynamics. We find that this thesis leads to a quantitative solution of the radiation problem. Finally we examine some of the implications of this thesis for the conception of causality.

THE STATUS OF RADIATIVE REACTION IN FIELD THEORY

A charged particle on being accelerated sends out electromagnetic energy and itself loses energy. This loss is interpreted as caused by a force acting on the particle given in magnitude and direction by the expression

$$\frac{2\,(\text{charge})^2\,(\text{time rate of change of acceleration})}{3\,(\text{velocity of light})^3},$$

when the particle is moving slowly, and by a more complicated expression when its speed is appreciable relative to the velocity of light. The existence of this force of radiative reaction is well attested: (a) by the electrical potential required to drive a wireless antenna; (b) by the loss of energy experienced by a charged particle which has been deflected, and therefore accelerated, in its passage near an atomic nucleus; and (c) by the cooling suffered by a glowing body.

The origin of the force of radiative reaction has not been nearly so clear as its existence.

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[3] W. Ritz, Ann. d. Chem. et d. Physique **13**, 145 (1908).
[4] H. Tetrode, Zeits. f. Physik **10**, 317 (1922).
[5] J. Frenkel, Zeits. f. Physik **32**, 518 (1925).
[6] J. L. Synge, Proc. Roy. Soc. London **A177**, 118 (1940).

Lorentz[7] considers the charged particle to have a finite size and attributes the force in question to the retarded action of one part of the particle on another. His expression for the force appears as a series in powers of the radius of the particle. The first term in the series gives the expression already mentioned. Otherwise, the derivation leads to difficulties:

(a) All higher terms depend explicitly upon the structure assumed for the entity. These dubious terms enter in a more and more important way into the calculated law of radiative reaction as the frequency of oscillation of the particle is raised, and the period approaches the time required for light to cross the system.

(b) Non-electric forces are required to hold together the charge distribution, according to Poincaré,[8] for to neglect such forces is to violate the relativistic relation between mass and energy. A composite system of this kind would possess an infinite number of internal degrees of freedom of oscillation. No consistent model has been found for the Lorentz electron in either classical or quantum mechanics.

Briefly, Lorentz attempts to propose a physical mechanism behind the radiative reaction, but arrives at a mathematically incomplete expression for this force.

Dirac,[9] in contrast, advances no explanation for the origin of the radiative damping, but supplies a well-defined and relativistically invariant prescription to calculate its magnitude:

Let the motion of the particle be given. Calculate the field produced by the particle from Maxwell's equations, with the boundary condition that at large distances from the particle this field shall contain only outgoing waves. In addition to the so-defined retarded field of the particle, calculate its advanced field, the sole change being the existence of only convergent waves at large distances. Define half the difference between retarded and advanced fields as the radiation field (half the quantity denoted as radiation field by Dirac). This field is everywhere finite. Evaluate it at the position of the particle and multiply by the magnitude of the charge to obtain the force of radiative reaction.

Dirac's prescription is appealing. (a) It is well-defined. (b) The calculated force reduces for slowly moving particles to the simple expression which was given above and which has been well-tested at non-relativistic velocities. (c) The calculation treats the elementary charge as being localized at a mathematical point, a picture which is not only physically reasonable but also translatable into quantum mechanics. (d) The elements of the prescription involve no more than standard electromagnetic theory plus the assumption that the radiation field, as above defined, is the source of the force.

The physical origin of Dirac's radiation field is nevertheless not clear. (a) This field is defined for times before as well as after the moment of acceleration of the particle. (b) The field has no singularity at the position of the particle and by Maxwell's equations must, therefore, be attributed either to sources other than the charge itself or to radiation coming in from an infinite distance.

We accept as reasonable Dirac's results. His concept of radiation field, however, we cannot adopt as an assumption subject to no further analysis. To do so would be to add to field theory a principle incapable of translation into the language of action at a distance.

To carry the analysis further requires us to find a new idea. We go back to a suggestion once made by Tetrode.[10] He proposed to abandon the conception of electromagnetic radiation as an elementary process and to interpret it as a consequence of an interaction between a source and an absorber. In his words,

"The sun would not radiate if it were alone in space and no other bodies could absorb its radiation. . . . If for example I observed through my telescope yesterday evening that star which let us say is 100 light years away, then not only did I know that the light which it allowed to reach my eye was emitted 100 years ago, but also the star or individual atoms of it knew already 100 years ago that I, who then did not even exist, would view it yesterday evening at such and such a time. . . . One might accordingly adopt the opinion that the amount of material in the universe determines the rate of emission. Still this is not necessarily so, for two competing absorption centers

[7] H. A. Lorentz (1892), republished in his *Collected Papers*, Vol. II, pp. 281 and 343. See also his treatise *The Theory of Electrons* (Leipzig, 1909), pp. 49 and 253.

[8] H. Poincaré, Rend. Palermo **21**, 165 (1906).

[9] P. A. M. Dirac, Proc. Roy. Soc. London **A167**, 148 (1938).

[10] H. Tetrode, Zeits. f. Physik **10**, 317 (1922). When we gave a preliminary account of the considerations which appear in this paper (Cambridge meeting of the American Physical Society, February 21, 1941, Phys. Rev. **59**, 683 (1941)) we had not seen Tetrode's paper. We are indebted to Professor Einstein for bringing to our attention the ideas of Tetrode and also of Ritz, who is cited in this article. An idea similar to that of Tetrode was subsequently proposed by G. N. Lewis, Nat. Acad. Sci. Proc. **12**, 22 (1926): "I am going to make the . . . assumption that an atom never emits light except to another atom, and to claim that it is as absurd to think of light emitted by one atom regardless of the existence of a receiving atom as it would be to think of an atom absorbing light without the existence of light to be absorbed. I propose to eliminate the idea of mere emission of light and substitute the idea of *transmission*, or a process of exchange of energy between two definite atoms or molecules." Lewis went nearly as far as it is possible to go without explicitly recognizing the importance of other absorbing matter in the system, a point touched upon by Tetrode, and shown below to be essential for the existence of the normal radiative mechanism.

will not collaborate but will presumably interfere with each other. If only the amount of matter is great enough and is distributed to some extent in all directions, further additions to it may well be without influence."

Tetrode's idea that the absorber may be an essential element in the mechanism of radiation has been neglected, perhaps partly because it appears to conflict with customary notions of causality, and partly also because of his mistaken belief that the new point of view could by itself explain quantum phenomena. In this connection he assumed that the interaction between charged particles should be described by forces more complicated than those given by electromagnetic theory. Finally, as Tetrode remarks, "on the last pages we have let our conjectures go rather far beyond what has mathematically been proven."

ABSORBER RESPONSE AS THE MECHANISM OF RADIATIVE REACTION

We take up the proposal of Tetrode that the absorber may be an essential element in the mechanism of radiation. Using the language of the theory of action at a distance, we give the idea the following definite formulation:

(1) An accelerated point charge in otherwise charge-free space does not radiate electromagnetic energy.

(2) The fields which act on a given particle arise only from other particles.

(3) These fields are represented by one-half the retarded plus one-half the advanced Liénard-Wiechert solutions of Maxwell's equations. This law of force is symmetric with respect to past and future. In connection with this assumption we may recall an inconclusive but illuminating discussion carried on by Ritz and Einstein in 1909, in which "Ritz treats the limitation to retarded potentials as one of the foundations of the second law of thermodynamics, while Einstein believes that the irreversibility of radiation depends exclusively on considerations of probability."[11] Tetrode, himself, like Ritz, was willing to assume elementary interactions which were not symmetric in time. However, complete reversibility is assumed here because it is an essential element in a unified theory of action at a distance. In proceeding on the basis of this symmetrical law of interaction, we shall be testing not only Tetrode's idea of absorber reaction, but also Einstein's view that the one-sidedness of the force of radiative reaction is a purely statistical phenomenon. This point leads to our final assumption:

(4) Sufficiently many particles are present to absorb completely the radiation given off by the source.

On the basis of these assumptions we shall consider as the source of radiation an accelerated charge located in the absorbing system. A disturbance travels outward from the source. By it each particle of the absorber is set in motion and caused to generate a field, half-advanced and half-retarded. The sum of the advanced effects of all particles of the absorber, evaluated in the neighborhood of the source, gives a field which we find to have the following properties:

(1) It is independent of the properties of the absorbing medium.

(2) It is completely determined by the motion of the source.

(3) It exerts on the source a force which is finite, is simultaneous with the moment of acceleration, and is just sufficient in magnitude and direction to take away from the source the energy which later shows up in the surrounding particles.

(4) It is equal in magnitude to one-half the retarded field minus one-half the advanced field generated by the accelerated charge. In other words, the absorber is the physical origin of Dirac's radiation field.

(5) This field combines with the half-retarded, half-advanced field of the source to give for the total disturbance the full retarded field which accords with experience.

It will be sufficient to establish these results in order to have both in field theory and in the theory of action at a distance a solution of the problem of radiation, including an explanation of the force of radiative damping.

We shall present four derivations of the reaction of radiation on the source of successively increasing generality. In the first we consider an absorber in which the particles are far from one another. We assume without proof that the disturbance which passes through the medium is the full retarded field of experience. In the second derivation we examine the field of the absorber in the neighborhood of the source and find it just such as to compensate the advanced field of the accelerated charge and to give a retarded field of the previously assumed magnitude. In this case we have allowed the medium to have arbitrary density. The third derivation —in contrast to the first two, where the source was taken to be at rest or moving only slowly— considers the case of motion with arbitrary velocity and leads to the same relativistic expression which Dirac has given for the force of radiative reaction. All three treatments proceed by adding up the fields owing to the individual particles of the absorber. A fourth derivation uses a much more general approach, assuming only that the medium is a complete absorber.

[11] W. Ritz and A. Einstein, Physik. Zeits. 10, 323 (1909); see also W. Ritz, Ann. d. Chemie et d. Physique 13, 145 (1908).

THE RADIATIVE REACTION: DERIVATION I

For a first analysis of the mechanism of radiative reaction, we shall simplify as much as possible the properties of the absorber:

(a) it is taken to be composed of free-charged particles;
(b) these corpuscles are at rest or are moving only slowly with respect to the particle which we treat as the source;
(c) the charged entities are well separated from one another;
(d) the particles occupy space to distances sufficiently great to bring about essentially complete absorption of radiation from the source.

We begin by considering the reaction set up between the source and a typical charge in the absorber when the particle of the source receives an acceleration $\mathfrak{A}$, by collision with a third particle or otherwise. The source has a charge $+e$ and, therefore, sends out an electromagnetic disturbance. This effect traverses the distance r_k to the particle of the absorber, reaching it at a time (r_k/c) seconds later than the instant of acceleration. For the electric field acting on the absorber at this place and time, we adopt the usual retarded solution of Maxwell's equation, in conformity with experience, but without any attempt in this first derivation of the force of radiative reaction to reconcile such an assumption with the half-retarded, half-advanced field of the theory of action at a distance. At the distances in which we are interested, the retarded field of the source reduces to the well-known expression,

$$-(e\mathfrak{A}/r_kc^2)\sin(\mathfrak{A}, r_k), \tag{1}$$

together with a term of electrostatic origin. This second term falls off inversely as the square of the distance and may, therefore, be neglected. The electric vector lies in the plane defined by the directions of $\mathfrak{A}$ and r_k, is perpendicular to r_k, and is considered positive when its component along the direction of $\mathfrak{A}$ is positive.

The typical particle of the absorber has a charge e_k and mass m_k. It will experience in the electric field of the disturbance an acceleration, $\mathfrak{A}_k$, equal to (e_k/m_k) times expression (1). Its motion will generate a field which will be half-advanced and half-retarded. The advanced part of this field will exert on the source a force simultaneous with the original acceleration. The component of this reactive force along the direction of the acceleration will be

$$-e(e_k\mathfrak{A}_k/2r_kc^2)\sin(\mathfrak{A}, r_k) = (\mathfrak{A}e^2/2c^4)(e_k^2/m_kr_k^2)\sin^2(\mathfrak{A}, r_k). \tag{2}$$

From expression (2) for the reactive force due to one particle of the absorber, we can evaluate the total effect due to many particles, present to the number N per unit of volume. The number of particles in a spherical shell of thickness dr_k will be $4\pi Nr_k^2dr_k$. For the particles in this shell the average value of the geometrical factor $\sin^2(\mathfrak{A}, r_k)$ will be $(2/3)$. Consequently we obtain for total force of reaction the integral of the expression

$$(2\mathfrak{A}e^2/3c^3)(2\pi Ne_k^2/m_kc)dr_k. \tag{3}$$

The force (3) gives an account of the phenomenon of radiative reaction which is not in accord with experience:

(1) The force acts on the source in phase with its acceleration; or in other words, it is proportional to the acceleration itself rather than to the time rate of change of acceleration.
(2) The reaction depends upon the nature of the absorbing particles.
(3) The force appears at first sight to grow without limit as the number of particles or the thickness of the absorber is indefinitely increased.

Nevertheless, proper addition of the effects due to all the particles of a complete absorber, with due allowance for their phase relations, does lead, as we shall see, to a reasonable expression for the reaction on the source.

There exists a phase lag between outgoing disturbance and returning reaction which we have not taken into account. The advanced force acting on the source due to the motion of a typical particle of the absorber is an elementary interaction between two charges, propagated with the speed of light in vacuum. On the other hand, the disturbance which travels outward from the source and determines the motion of the particle in question is made up not only of the proper field of the originally accelerated charge, but also of the secondary fields generated in the material of the absorber. The elementary interactions are of course propagated with the speed of light; but the combined disturbance travels, as is well known from the theory of the refractive index, at a different speed,

$$c/(\text{refractive index}) = c/n.$$

In order to speak of the change in velocity of the disturbance, or to treat the refractive index of the absorber in a well-defined way, it will be necessary to consider a single Fourier component of the acceleration. The connection between acceleration and reactive force being a linear one, it will be legitimate to decompose the acceleration into parts of this kind, and later to recompose the corresponding Fourier components of the radiative reaction. We shall, therefore, suppose for the moment that the primary acceleration varies with time according to the formula

$$\mathfrak{A}=\mathfrak{A}_0 \exp{(-i\omega t)}, \tag{4}$$

where ω represents the circular frequency of the motion. A disturbance of this frequency will experience in a medium of low density a refractive index given by the familiar formula,

$$n=1-2\pi Ne_k^2/m_k\omega^2. \tag{5}$$

Thus the radiative reaction which reaches the source from a depth r_k in the absorber will lag in phase behind the acceleration by the angle

$$\omega(r_k/c-nr_k/c)=(2\pi Ne_k^2/m_kc\omega). \tag{6}$$

We apply this phase correction to the contribution (3) of absorber particles in the range r_k to r_k+dr_k, and sum over all depths in the medium to obtain the total reactive force,

$$(2e^2/3c^3)\mathfrak{A}\int_0^\infty (2\pi Ne_k^2/m_kc)dr_k \times\exp{(-ir_k 2\pi Ne_k^2/m_kc\omega)}. \tag{7}$$

This integral will converge at the upper limit when we allow for the existence of a small but finite coefficient of absorption in the medium. Or in the language of physical optics, so familiar from the writings of R. W. Wood, we can say that we have to determine the combined effect of a number of wave zones, alternately in and out of phase with the acceleration. The resultant force is 90° out of phase with the acceleration and is equal in magnitude to the arithmetic sum of the contributions from depths up to a point where the phase lag is one radian:

$$\begin{aligned}(\text{total reaction})&=(2e^2/3c^3)(-i\omega\mathfrak{A})\\&=(2e^2/3c^3)(d\mathfrak{A}/dt).\end{aligned} \tag{8}$$

This result, derived by considering only a single Fourier component of the acceleration, no longer contains explicit reference to the frequency of that component. Consequently expression (8) applies whatever is the dependence of acceleration upon time, so long as the velocities of all particles remain non-relativistic. In this respect we have a quite general derivation of the law of radiative reaction generally accepted as correct for a slowly moving particle subjected to an arbitrary acceleration.

We conclude that the force of radiative reaction arises, not from the direct action of a particle upon itself, but from the advanced action upon this charge caused by the future motion of the particles of the absorber.

RADIATIVE REACTION: DERIVATION II

In the above treatment we considered first the retarded electromagnetic disturbance traveling outward in the absorbing medium; second, the motion of the particles of the medium due to this disturbance; third, the advanced part of the elementary fields produced by these motions; fourth, the sum of these fields at the position of the source. The same chain of reasoning will allow us to sum the elementary advanced fields of the particles of the absorbing medium at points in the neighborhood of the source. We shall find that this field is just sufficient, when added to the half-advanced, half-retarded field of the source itself, to give the usual full strength purely retarded field which one is accustomed to attribute to a radiating source. Thus we shall justify the assumption made in the first derivation as to the strength of the outgoing disturbance. In order to make it clear that our reasoning is not circular, we shall represent the magnitude of the disturbance by a multiple, (?), of the usual full retarded field, and shall actually deduce the value unity for this at present undetermined factor.

We shall now evaluate the contribution of particles in the absorber to the electric field acting in the region roundabout the source. In order to simplify the geometrical considerations as much as possible, we shall visualize the source as located at the center of a spherical cavity of radius R in the medium. We shall take the distance, r, from the source to the point of evaluation of the field to be small in comparison with this radius. We shall however give up the assumption that the particles of the absorber are necessarily free, or that they are far from one another. To make this generalization in our previous derivation, we shall express the acceleration of the typical particle of the absorber for a disturbance of circular frequency ω in the form

$$(\text{electric field of disturbance})\cdot(e_k/m_k)\cdot p(\omega). \quad (9)$$

Here $p(\omega)$ is in general a complex function of ω which approaches unity only in the case of weak binding or high frequencies. The factor $p(\omega)$, according to the theory of dispersion, determines the complex refractive index, $n-ik$, of the medium:

$$1-(n-ik)^2=(4\pi Ne_k^2/m_k\omega^2)p(\omega). \quad (10)$$

The advanced field produced by the absorber at the distance from the source will be given in amplitude and phase by the product of the following factors:

$\mathfrak{A}=\mathfrak{A}_0 \exp(-i\omega t)$,
: the acceleration of the source, here assumed for simplicity to be periodic, although this periodicity will drop out of the final result.

$-(e/r_kc^2)\sin(\mathfrak{A}, r_k)$,
: the factor by which the acceleration must be multiplied to obtain the strength of the full retarded electric field in vacuum at a great distance, r_k, from an accelerated particle of charge e.

(?),
: factor as yet undetermined, which allows for the possibility that the disturbance which is propagated outward from the particle, and which is in general due only partly to the source itself, may differ in strength from the usual full retarded field. For an isolated charge in otherwise charge-free space this factor is equal to $(\frac{1}{2})$. In the present case of a complete absorber we shall however later find for this factor the value unity. The product of the factors so far gives the strength of the electric field which would act on an isolated particle at the distance r_k.

$\exp(i\omega r_k/c)$,
: the phase of the disturbance which would act on such an isolated particle.

$2(1+n-ik)^{-1}$,
: factor by which the strength of the electric field of the disturbance inside the medium is reduced by reflection at the wall of the cavity, a factor taken over from electromagnetic theory.

$\exp(i\omega(n-ik-1)(r_k-R)/c)$,
: factor allowing for the change in phase and amplitude of the disturbance produced by propagation to the depth (r_k-R) in the medium.

$(e_k/m_k)p(\omega)$
: factor relating acceleration of absorber particle to electric field experienced by it.

$-(e_k/2r_kc^2)\sin(\mathfrak{A}, r_k)$,
: factor to be multiplied by acceleration of absorber particle to give the magnitude of the component of the advanced electric field produced by the absorber in the neighborhood of, and parallel to the acceleration of, the source.

$\exp(-i\omega r_k/c)$,
: factor allowing for the difference in phase between (a) the advanced field of the absorber as evaluated at the source itself and (b) the acceleration of the typical absorber particle.

$\exp(i\omega r\cos(r, r_k)/c)$,
: correction to be applied to phase of absorber field at the source itself in order to evaluate this field at the distance, r, from the source. The product of the factors so far gives in magnitude and phase the advanced field at this point owing to a single particle of the absorber.

$Nr_k^2dr_kd\Omega$,
: number of absorber particles in the element of solid angle $d\Omega$ and in the interval of distance dr_k.

We evaluate the product of the listed factors and sum over all particles of the absorber to evaluate the total advanced field of the absorber in the neighborhood of the source:

$$(?)(e/c^3)\mathfrak{A}_0\exp(-i\omega t)\int_\Omega \exp(i(\omega/c)r\cos(r, r_k)) \times\sin^2(\mathfrak{A}, r_k)(d\Omega/4\pi)$$
$$\int_R^\infty (4\pi Ne_k^2/m_kc)p(\omega)(1+n-ik)^{-1}dr_k \times\exp(i\omega(n-ik-1)(r_k-R)/c). \quad (11)$$

The last integral is simplified by the relationship (10) between refractive index and physical properties of the medium to an expression

$$\int_R^\infty (\omega^2/c)(1-n+ik)dr_k \times\exp(i\omega(n-ik-1)(r_k-R)/c)=-i\omega, \quad (12)$$

completely independent of the properties of the absorber. Having thus summed over all particles lying in a given direction, we sum over different directions, using the relations

$$(d\Omega/4\pi)=(1/2)d\cos(r, r_k)(d\varphi/2\pi), \quad (13)$$

where φ is the dihedral angle between the $(r, \mathfrak{A})$ and (r, r_k) planes; and

$$\int \sin^2(\mathfrak{A}, r_k)(d\varphi/2\pi)$$
$$=(2/3)\int[1-P_2(\cos(\mathfrak{A}, r_k))]d\varphi/2\pi$$
$$=(2/3)[1-P_2(\cos(\mathfrak{A}, r))P_2(\cos(r_k, r))]. \quad (14)$$

Also we note the integrals

$$(1/2)\int_{-1}^{1}\exp(iu\cos\theta)d(\cos\theta)=F_0(u)$$

$$=u^{-1}\sin u\simeq\begin{cases}1\text{ for small }u\\(e^{iu}-e^{-iu})/2iu\text{ for all }u,\end{cases}\quad(15)$$

and

$$(1/2)\int_{-1}^{1}\exp(iu\cos\theta)P_2(\cos\theta)d(\cos\theta)=F_2(u)$$

$$=(u^{-1}-3u^{-3})\sin u+3u^{-2}\cos u$$

$$\simeq\begin{cases}-u^2/15\text{ for small }u\\(e^{iu}-e^{-iu})/2iu\text{ for large }u.\end{cases}\quad(16)$$

By use of these mathematical results, we find that the advanced field of the absorber has in the cavity an electric component parallel to the acceleration of the source which is given in magnitude and phase by the expression

$$(?)(2e/3c^3)(-i\omega\mathfrak{A}_0)\exp(-i\omega t)$$
$$[F_0(\omega r/c)-P_2(\cos(\mathfrak{A},r))F_2(\omega r/c)].\quad(17)$$

The radiation field so obtained reduces at the source itself to the form

$$(?)(2e/3c^3)(d\mathfrak{A}/dt),\quad(18)$$

and at distances a number of wave-lengths from the source goes over into the expression

$$-(?)(e\mathfrak{A}_0/2rc^2)\exp(i\omega r/c-i\omega t)$$
$$+(?)(e\mathfrak{A}_0/2rc^2)\exp(-i\omega r/c-i\omega t).\quad(19)$$

In words, formula (19) states that the advanced field of the absorber is equal in the neighborhood of the accelerated particle to the still undetermined factor (?), multiplied by the difference between half the retarded field (first term) and half the advanced field (second term) which one calculates for the source itself.

It is instructive to see how superposition of the advanced fields of a large number of particles can give the appearance of both retarded and advanced fields due to the source itself. The advanced field of a single charge of the absorber can be symbolized as a sphere which is converging towards the particle and which will collapse upon it at just the moment when it is disturbed by the source. But at the moment when the source particle itself was accelerated, the sphere in question had a substantial radius. One point on it touched, or nearly touched, the source. The shrinking sphere therefore appears to the source as a nearly plane wave which passes over it headed towards one of the particles of the absorber. When we consider the effect of all the absorbing charges, we have to visualize an array of approximately plane waves, all marching towards the source and passing over it in step. The resultant of these individual effects is a spherical wave, the envelope of the many nearly plane waves. The sphere converges, collapses on the source, and then pours out again as a divergent sphere. An observer in the neighborhood will gain the impression that this divergent wave originated from the source.

A test particle will be unable to make a separation between the two retarded fields, one properly owing to the source, the other really owing to the advanced field of the absorber. Thus we have for the disturbance diverging from the source the relation

$$\begin{pmatrix}\text{total disturbance}\\\text{diverging from}\\\text{source}\end{pmatrix}=\begin{pmatrix}\text{proper retarded}\\\text{field of source}\\\text{itself}\end{pmatrix}+\begin{pmatrix}\text{field apparently diverging from source,}\\\text{actually composed of parts converging}\\\text{on individual absorber particles}\end{pmatrix}.\quad(20)$$

We are now in a position to evaluate the undetermined factor, (?), in the expression we have used in the above analysis for the force acting on the typical particle of the absorber. We have only to express all three terms of Eq. (20) in units of the usual retarded solution of Maxwell's equations, a solution which asymptotically for large distances from the source gives for the electric field parallel to the acceleration the expression

$$-(e\mathfrak{A}/rc^2)\sin^2(\mathfrak{A},r).\quad(21)$$

To evaluate the third term in (20), we refer back to Eq. (19). Thus we find the algebraic equation

$$(?)=(1/2)+(?/2),\quad(22)$$

of which the solution is

$$(1)=(1/2)+(1/2).\quad(23)$$

From our derivation we find for the disturbance diverging from an accelerated charge the full retarded field required by experience.

Along the same lines we can find the strength of the advanced field converging upon the source before the moment of acceleration:

$$\begin{pmatrix}\text{total disturbance}\\ \text{converging on}\\ \text{source}\end{pmatrix}=\begin{pmatrix}\text{proper advanced}\\ \text{field of source}\\ \text{itself}\end{pmatrix}+\begin{pmatrix}\text{field apparently convergent on source,}\\ \text{actually composed of parts convergent}\\ \text{on individual absorber particles}\end{pmatrix}. \tag{24}$$

At distances of several wave-lengths from the source, the two terms on the right possess simple mathematical expressions. Measured in terms of (21) as a unit of field strength, the right-hand side of (24) has the value $(1/2)-(1/2)=0$. We conclude that there is no net disturbance converging upon the source prior to the time of acceleration. The advanced field of the source is completely compensated by the advanced field of the absorber.

Our picture of the mechanism of radiation is seen to be self-consistent. Any particle on being accelerated generates a field which is half-advanced and half-retarded. From the source a disturbance travels outward into the surrounding absorbing medium and sets into motion all the constituent particles. They generate a field which is equal to half the retarded minus half the advanced field of the source. In this field we have the explanation of the radiation field assumed by Dirac. The radiation field combines with the field of the source itself to produce the usual retarded effects which we expect from observation, and such retarded effects only. The radiation field also acts on the source itself to produce the force of radiative reaction. What we have said of one particle holds for every particle in a completely absorbing medium. All advanced fields are concealed by interference. Their effects show up directly only in the force of radiative reaction. Otherwise we appear to have a system of particles acting on each other via purely retarded forces.

RADIATIVE REACTION: DERIVATION III

So far the source has been assumed to be at rest, or in slow motion, at the time of acceleration. The expression derived above for the force of radiative reaction is therefore limited in its applicability. To obtain the corresponding law of damping for a swift particle three possibilities suggest themselves, each calling for a mathematical technique quite different from that of the other two. The first and simplest procedure is to look at the particle from a frame of reference moving with nearly its own speed, apply in this frame the expression which we already have, and then transform back to the laboratory frame of reference. This application of the transformation of Lorentz is perfectly legitimate but not especially instructive.

A second method to calculate the force of reaction for a fast particle comes from Dirac. He makes the assumption that the damping arises from the action on the particle of a field equal to half the difference between the particle's own retarded and advanced fields, a conception which we have now interpreted in terms of the radiative reaction of the absorber. As each of the two fields is individually singular at the location of the charge, evaluation of the difference requires one to apply a limiting process which presents a certain mathematical difficulty, though in principle perfectly straightforward.

In connection with the limiting process of Dirac, it is interesting to refer back to the calculation of radiative reaction made by Lorentz on the model of an extended charge, every part of which exerted a retarded effect upon every other part. The elementary retarded field can be written in the form

$$\begin{pmatrix}\text{retarded}\\ \text{field}\end{pmatrix}=\left[\tfrac{1}{2}\begin{pmatrix}\text{retarded}\\ \text{field}\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}\text{advanced}\\ \text{field}\end{pmatrix}\right]+\left[\tfrac{1}{2}\begin{pmatrix}\text{retarded}\\ \text{field}\end{pmatrix}-\tfrac{1}{2}\begin{pmatrix}\text{advanced}\\ \text{field}\end{pmatrix}\right].$$

Here the first term is singular and is related to what Lorentz called the electromagnetic mass of the particle. The second part, on the other hand, is the only one asymmetrical in time and capable of contributing to the force of radiative reaction. In present terms, the procedure of Lorentz amounts to an ingenious means to determine the limiting value of Dirac's radiation field at the position of the source. Unfortunately the procedure is not convenient to apply to a rapidly moving extended charge because of the relativistic contraction of its spherical form.

The third procedure to evaluate the law of radiative damping for a swift particle is to calculate directly the reaction of the absorber on the source, along the lines of derivations I and II. This approach uses the expression for the field of a radiating charge, not at small distances, where it is singular, but at large distances, where it has a simple asymptotic form. We shall explore this type of derivation because of its direct relation to the absorber theory of radiation.

We idealize the absorber as before as a sphere of very large radius, r, centered on the point reached by the given particle at a chosen instant. At the surface of the absorber, those constituents of the field which drop away as $1/r^2$ will have become negligible in comparison with those which fall off as $1/r$. The typical particle on this surface experiences an electric field perpendicular to the direction of r. The magnitude of this field was represented in the case of a slowly moving source by an expression of the form, $-(e/rc^2)$(component of acceleration perpendicular to r), and is similarly representable in the present case by an expression of the type, $-(e/r)$(function of motion)$_\Omega$. Here the function in parenthesis, while more complicated than before, still depends only on the motion of the particle and the direction to the point of action. The influence of this disturbance causes the particles of the absorber to generate a field, the advanced part of which at the position of the source claims our attention. We consider the portion of this returned field arising from particles of the absorber which lie within an element of solid angle $d\Omega$. The electric component of this field is perpendicular to r and had for a slowly moving source the magnitude $(e/c^2)(-i\omega/c)$(component of acceleration perpendicular to r)$(d\Omega/4\pi)$ when the acceleration was a periodic function of time, and more generally was given by the derivative $(e/c^2)(d/cdt)$ $\times$(component of acceleration perpendicular to r) $\times(d\Omega/4\pi)$.

The relationship between returned field and original disturbance is a property only of the absorber and is independent of the state of motion of the source. Consequently, for the case of a particle moving at arbitrary velocity the returned electric field is perpendicular to r and equal in magnitude to

$$e(d/cdt)(\text{function of motion})_\Omega(d\Omega/4\pi).$$

What we have said of the electric field applies also to the magnetic field, because at great distances from an accelerated particle the two vectors have equal magnitudes and perpendicular directions. Thus we conclude that the reaction of the absorber on the source is described by a field, F_{mn}, which is directly related to the retarded field, R_{mn}, of the source at great distances, r, by the equation[12]

$$F_{mn}=-r\int(\partial R_{mn}/\partial x^4)(d\Omega/4\pi). \qquad (25)$$

The retarded field of the source particles, R_{mn}, in Eq. (25) is derived from the retarded potentials

$$A_m=2e\int \dot{a}^m(\alpha)\delta(xa_\mu xa^\mu)d\alpha,$$

through the equation

$$R_{mn}=(\partial A_n/\partial x^m)-(\partial A_m/\partial x^n).$$

Here the integration over the proper cotime, α, goes only over that portion of the world line of particle a from which a retarded disturbance can reach the point of action, x^m. The significant value of α is connected with the coordinates x^m

[12] Here and below we use the following notation:

$\left.\begin{matrix}x^1=x_1\\x^2=x_2\\x^3=x_3\end{matrix}\right\}$ the three space coordinates of a typical point of evaluation of the field.

$x^4=-x_4$, a quantity also having the dimensions of a length, and given by the product of the velocity of light and the time elapsed between a certain zero hour and the moment of observation.

a^m, similar space-time coordinates of a typical point on the world line of the ath particle.

Successive points along the world line are designated by the values of a parameter, α, the proper cotime, which has the dimensions of a length and is equal to the product of the velocity of light and the proper time. The difference, $d\alpha$, in proper cotime between two neighboring points has the same sign as the difference da^4, and is given in magnitude by the equation

$$(d\alpha)^2=c^2(\text{time interval})^2-(\text{space interval})^2=-da_\mu da^\mu.$$

Derivatives with respect to α are denoted by dots. In comparing formulae given in this notation with those given elsewhere in the literature, it will be noted that some authors go from contravariant to covariant representation of a vector by reversing the sign of its space components and leaving its time component unaltered; also that dots are often used to indicate differentiation with respect to proper time, rather than proper cotime. In our notation the derivatives $\dot{a}^m$ are dimensionless quantities which satisfy the relation $\dot{a}_\mu\dot{a}^\mu=-1$. We use xa^m as an abbreviation for the vector, x^m-a^m. The usual scalar potential of the electromagnetic field is represented by the component A^4 of a four-vector, of which the other three parts, A^1, A^2, A^3, constitute the space components of the customary vector potential. The typical component of the field is given in the equation $F_{mn}=(\partial A_n/\partial x^m)-(\partial A_m/\partial x^n)$, where we have for the electric field $E_x=F_{14}=-F_{41}$, etc., and for the magnetic field $H_x=F_{23}=-F_{32}$, etc.

by the equation

$$xa_\mu xa^\mu = 0. \tag{26}$$

The integration yields

$$A_m = -e\dot{a}_m/(\dot{a}_\mu xa^\mu).$$

In differentiating this expression with respect to the coordinates of the point of observation, we have to allow for the associated change in the value of the proper cotime, given by the differential of (26),

$$(dx_\mu - \dot{a}_\mu d\alpha)(x^\mu - a^\mu) = 0,$$

or

$$d\alpha = (xa_\mu dx^\mu)/(\dot{a}_\mu xa^\mu). \tag{27}$$

Thus the retarded field of a is found to be given by the expression

$$R_{mn} = e(\dot{a}_\mu xa^\mu)^{-2}(\ddot{a}_m xa_n - \ddot{a}_n xa_m) + e(1+\ddot{a}_\mu xa^\mu)(\dot{a}_\mu xa^\mu)^{-3}(-\dot{a}_m xa_n + \dot{a}_n xa_m).$$

All terms in this expression fall off at large distances inversely as the first power of the separations xa, except for the terms arising from the unity in the factor $(1+\ddot{a}_\mu xa^\mu)$, which we may henceforth omit. For the same reason in differentiating the field with respect to x^4, we may treat all differences xa^m as constant. Thus we find in the limit of large distances

$$\begin{aligned}
-r\int &(\partial R_{mn}/\partial x^4)(d\Omega/4\pi) \\
&= r\int(-xa_4/\dot{a}_\mu xa^\mu)(d/d\alpha)R_{mn}(d\Omega/4\pi) \\
&= e\int\{3(\ddot{a}_\mu xa^\mu)(\dot{a}_\mu xa^\mu)^{-4}(xa_m\ddot{a}_n - xa_n\ddot{a}_m) \\
&\quad - (\dot{a}_\mu xa^\mu)^{-3}(xa_m\dddot{a}_n - xa_n\dddot{a}_m) \\
&\quad + (\dddot{a}_\mu xa^\mu)(\dot{a}_\mu xa^\mu)^{-4}(xa_m\dot{a}_n - xa_n\dot{a}_m) \\
&\quad - 3(\ddot{a}_\mu xa^\mu)^2(\dot{a}_\mu xa^\mu)^{-5}(xa_m\dot{a}_n - xa_n\dot{a}_m)\} \\
&\quad \times (r^2 d\Omega/4\pi).
\end{aligned} \tag{28}$$

As variables of integration it is convenient to use a colatitude θ and azimuthal angle φ, taking for polar axis the direction of the space component, $(\dot{a}^1, \dot{a}^2, \dot{a}^3)$, of the four-vector, $\dot{a}^m$. With this choice of variables the denominator of the typical term in the preceding expression is a power of the factor $(\dot{a}_\mu xa^\mu) = r(\dot{a}_4 + \dot{a}\cos\theta)$, where $\dot{a}_4{}^2 - \dot{a}^2 = 1$. The absence of the azimuthal angle from the denominator and the relatively simple form of the numerator makes it easy to carry out the integration over φ. The numerator of the typical term then reduces to a polynomial in $\cos\theta$. The integration over θ therefore leads only to algebraic functions of $\cos\theta$ to be evaluated at the two limits $\cos\theta = \pm 1$. The reduction of the resulting expressions to simple form requires rather long calculation. The final result for the field of radiative reaction at the location of the source is

$$\begin{aligned}
F_{mn} &= -r\int(\partial R_{mn}/\partial x^4)(d\Omega/4\pi) \\
&= (2e/3)(\dot{a}_m\dddot{a}_n - \dddot{a}_m\dot{a}_n).
\end{aligned} \tag{29}$$

This expression for the field of the absorbing particles agrees with that given by Dirac for half the difference of retarded and advanced fields due to the source itself, provided account is taken of the difference between the present notation and his.

If we define the force of radiative reaction through its contribution to the product of the mass of the particle by its acceleration, $mc^2\ddot{a}^m$, then we have for this force the expression

$$eF_{m\mu}\dot{a}^\mu = (2e^2/3)(\dot{a}_m\dddot{a}_\mu - \dddot{a}_m\dot{a}_\mu)\dot{a}^\mu. \tag{30}$$

In the case of a slowly moving particle the first space component of this force is readily evaluated by noting that (1) $\dot{a}_m$ is of the order of the ratio of the velocity of the corpuscle to the speed of light and is therefore negligible; (2) the quantity $-\dot{a}_\mu\dot{a}^\mu$ has the value unity; and (3) the derivative $\dddot{a}_m$ represents $(1/c^3)$ times the time rate of change of the given component of the acceleration, $\mathfrak{A}$. Consequently, the expression (30) reduces in the non-relativistic limit to the usual formula, $(2e^2/3c^3)(d\mathfrak{A}/dt)$, for the damping force.

From the properties of the retarded field at large distances from an accelerated particle in motion at an arbitrary velocity, we have obtained an expression for the force of radiative reaction previously derived by Dirac on the assumption that this force arises from half the difference of the advanced and retarded fields of the particle itself. It is, therefore, of interest to see that this equivalence can be demonstrated without going through the rather long calculations which are required on either method of derivation to obtain explicit expressions for the force of radiative reaction. To bring out the relationship between the two derivations, we go back to that expression for the retarded field of the source which contains a delta function, and arrange the evaluation of Eq. (25) in such a way as always to keep a delta function in evidence. Thus we write the retarded field in the form

$$R_{mn}=2e\int[-\dot{a}_m(\partial/\partial x^n)+\dot{a}_n(\partial/\partial x^m)]\delta(xa_\mu xa^\mu)d\alpha.$$

In order to postpone the differentiation of the delta function, we adopt an expedient to transform the variable of differentiation. We consider in addition to the actual world line of the source, $a^m(\alpha)$, a displaced world line, a particle moving along which reaches at the proper cotime, α, the point $\bar{a}^m(\alpha)=a^m(\alpha)+D^m$, where the D^m are four numbers independent of α. We note that the derivative with respect to x^m of any function of the differences $x^k-\bar{a}^k$ is equal to the negative of the derivative of the same function with respect to D^m. Consequently we may write the expression for the retarded field in the form

$$R_{mn}=2e\int[\dot{a}_m(\partial/\partial D^n)-\dot{a}_n(\partial/\partial D^m)]\times\delta(x\bar{a}^\mu x\bar{a}_\mu)d\alpha, \quad (31)$$

where the result is to be evaluated in the limit when the displacements D^m go to zero.

We now insert expression (31) for the retarded field into the integral for the field returned by the absorber,

$$F_{mn}=r\int(\partial R_{mn}/\partial D^4)(d\Omega/4\pi),$$

and encounter the integral

$$F_{mn}=2er(\partial/\partial D^4)\int\int[\dot{a}_m(\partial/\partial D^n)-\dot{a}_n(\partial/\partial D^m)]\times\delta(x\bar{a}_\mu x\bar{a}^\mu)d\alpha(d\Omega/4\pi).$$

To bring out the meaning of this integral, we note that we want the radiative reaction on the source at a definite point, $a^m=a^m(\alpha^*)$, along its world line; that this point is at the center of a sphere of radius r; and that advanced disturbances from the particles on the inner surface of this sphere contribute to the force at this point only if they start at a cotime, x^4, equal to $r+a^4(\alpha^*)$. Consequently, x^4 has this fixed value as the integration over the surface of the sphere is carried out. Also during this integration we keep fixed the variable α and consequently hold constant $\bar{a}=\bar{a}(\alpha)$. Under these circumstances it is convenient to adopt for variable of integration the angle θ between the space directions $a\bar{a}$ and ax:

$$(\bar{a}x)^2=(a\bar{a})^2+(ax)^2-2(a\bar{a})(ax)\cos\theta.$$

Then we have

$$\int\delta(\bar{a}x_\mu\bar{a}x^\mu)(d\Omega/4\pi)=(1/2)\int_{-1}^{1}\delta(\bar{a}x_\mu\bar{a}x^\mu)d(\cos\theta)$$

$$=\int_{(\bar{a}x)=(ax)-(a\bar{a})}^{(\bar{a}x)=(ax)+(a\bar{a})}\delta(\bar{a}x_\mu\bar{a}x^\mu)\times d[(\bar{a}x)^2-(a\bar{a})^2-(ax)^2]/4(a\bar{a})(ax)$$

$$=\int\delta(\bar{a}x_\mu\bar{a}x^\mu)d(\bar{a}x_\mu\bar{a}x^\mu)/4(a\bar{a})r.$$

In this last expression the range of integration includes the point, $\bar{a}x_\mu\bar{a}x^\mu$, for which the delta function gives a contribution, only if there are some points on the surface of the sphere which can be reached simultaneously by two retarded waves which start out with $a^m(\alpha^*)$ and $\bar{a}^m(\alpha)$ as centers. This condition will be satisfied if and only if $\bar{a}(\alpha)$ lies between the forward and backward light cones drawn with $a(\alpha^*)$ as origin. Thus we have

$$F_{mn}=e(\partial/\partial D^4)\int d\alpha[\dot{a}_m(\partial/\partial D^n)-\dot{a}_n(\partial/\partial D^m)]\times\begin{Bmatrix}1/2(a\bar{a}) & \text{when } \bar{a}a_\mu\bar{a}a^\mu>0\\ 0 & \text{when } \bar{a}a_\mu\bar{a}a^\mu<0\end{Bmatrix}.$$

The differentiation with respect to D^4 of the discontinuous function in the last pair of brackets gives a function which has the character of a delta function except for a change in sign at one of the singularities. Specifically, writing

$$\delta(\bar{a}a_\mu\bar{a}a^\mu)=\delta_+ +\delta_-,$$

where δ_+ is different from zero only when a retarded disturbance from $\bar{a}(\alpha)$ can reach the point $a(\alpha^*)$, and δ_- is different from zero only when an advanced disturbance from $\bar{a}(\alpha)$ can reach the point $a(\alpha^*)$, we have

$$(\partial/\partial D^4)\begin{Bmatrix}1/2(a\bar{a}) & \text{when } \bar{a}a_\mu\bar{a}a^\mu>0\\ 0 & \text{when } \bar{a}a_\mu\bar{a}a^\mu<0\end{Bmatrix}=\delta_+-\delta_-.$$

Then the field due to the absorber takes the form

$$F_{mn}=e\int[\dot{a}_m(\partial/\partial D^n)-\dot{a}_n(\partial/\partial D^m)](\delta_+-\delta_-)d\alpha$$

$$=e\int[\dot{a}_n(\partial/\partial a^m)-\dot{a}_m(\partial/\partial a^n)](\delta_+-\delta_-)d\alpha. \quad (32)$$

In other words, the reactive field at the point $a^m = a^m(\alpha^*)$ of the actual path is equal to half the retarded minus half the advanced field due to an equal charge moving on a world line of identical shape, all points of which are displaced by the amount D^m, this field evaluated in the limit $D^m \to 0$. This result establishes the connection between two different methods of evaluating the force of radiative reaction, one based on the properties of the retarded field of the source at great distances, the other containing half the difference of retarded and advanced fields at the location of the source itself.

THE RADIATIVE REACTION: DERIVATION IV

From the preceding applications of the absorber theory of radiation, it has become clear that such properties of the absorber as refractive index and density have no bearing on the magnitude of the force of radiative reaction. The only essential point is that the medium should be a complete absorber. We therefore expect that there should somehow be a means to take this point into account in a very general way.

In physical terms, complete absorption implies that a test charge placed anywhere outside the absorbing medium will experience no disturbance. In mathematical terms, using $\overset{(k)}{F}_{\text{ret}}$ and $\overset{(k)}{F}_{\text{adv}}$ to denote the retarded and advanced fields due to the kth particle, we have

$$\textstyle\sum_k(\frac{1}{2}\overset{(k)}{F}_{\text{ret}}+\frac{1}{2}\overset{(k)}{F}_{\text{adv}})=0 \text{ (outside the absorber)}. \quad (33)$$

From the fact that this sum vanishes outside the absorber everywhere and at all times, it follows that each of the two sums also vanishes outside the absorber:

$$\textstyle\sum_k \overset{(k)}{F}_{\text{ret}}=0 \text{ (outside)} \quad (34)$$

and

$$\textstyle\sum_k \overset{(k)}{F}_{\text{adv}}=0 \text{ (outside)}. \quad (35)$$

Thus, the one sum, if it does not vanish, represents at large distances an outgoing wave, and the other represents a converging wave; but complete destructive interference between two such waves is impossible. Hence, if their sum vanishes, so does each field individually. From this conclusion it follows that the difference of the fields vanishes outside the absorber at all times:

$$\textstyle\sum_k(\frac{1}{2}\overset{(k)}{F}_{\text{ret}}-\frac{1}{2}\overset{(k)}{F}_{\text{adv}})=0 \text{ (outside)}. \quad (36)$$

The field (36), in contrast to the fields (33)–(35), has no singularities within the absorber; it is a solution of Maxwell's equations for free space. Vanishing outside the absorber at all times, it must therefore forever be zero inside. The special property of a completely absorbing medium is expressed by the equation

$$\textstyle\sum_k(\overset{(k)}{F}_{\text{ret}}-\overset{(k)}{F}_{\text{adv}})=0 \text{ (everywhere)}. \quad (37)$$

The consequences of Eq. (37) for the force on a typical particle are easily deduced. On the ath charge the entire field acting is given, according to the theory of action at a distance, by the sum

$$\sum_{k\neq a}(\tfrac{1}{2}\overset{(k)}{F}_{\text{ret}}+\tfrac{1}{2}\overset{(k)}{F}_{\text{adv}}). \quad (38)$$

This expression can be broken down into three parts:

$$\sum_{k\neq a}\overset{(k)}{F}_{\text{ret}}+(\tfrac{1}{2}\overset{(a)}{F}_{\text{ret}}-\tfrac{1}{2}\overset{(a)}{F}_{\text{adv}})-\sum_{\text{all }k}(\tfrac{1}{2}\overset{(k)}{F}_{\text{ret}}-\tfrac{1}{2}\overset{(k)}{F}_{\text{adv}}). \quad (39)$$

Of these terms the third has just been shown to vanish for a complete absorber. The second gives rise to the phenomenon of radiative damping. In the case of non-relativistic velocities we have the result

$$e_a(\tfrac{1}{2}\overset{(a)}{E}_{\text{ret}}-\tfrac{1}{2}\overset{(a)}{E}_{\text{adv}})=(2e_a^2/3c^3)(d\mathfrak{A}_a/dt); \quad (40)$$

and in the case of swift particles we have for the force on the ath charge

$$e_a(\tfrac{1}{2}\overset{(a)}{F}_{n\alpha\ \text{ret}}-\tfrac{1}{2}\overset{(a)}{F}_{n\alpha\ \text{adv}})\dot{a}^\alpha.$$

This expression reduces, according to Dirac, to the form

$$(2e_a^2/3)(\dot{a}_n\dddot{a}_\alpha-\dddot{a}_n\dot{a}_\alpha)\dot{a}^\alpha, \quad (41)$$

in agreement with the reaction of the absorber as calculated in the preceding derivation. With this reactive term and the first term of (39), we arrive at the equation of motion of the typical particle in a completely absorbing medium

$$m_a\ddot{a}_n=e_a\sum_{k\neq a}\overset{(k)}{F}_{n\alpha\ \text{ret}}\dot{a}^\alpha$$
$$+(2e_a^2/3)(\dot{a}_n\dddot{a}_\alpha-\dddot{a}_n\dot{a}_\alpha)\dot{a}^\alpha. \quad (42)$$

In arriving at this equation we have shown that the half-advanced, half-retarded fields of the theory of action at a distance lead to a satisfactory account of the mechanism of radiative reaction and to a description of the action of one particle on another in which no evidence of the advanced fields is apparent. We find in the case of an absorbing universe a complete equivalence between the theory of Schwarzschild and Fokker on the one hand and the usual formalism of electrodynamics on the other. This is what was to be proved.

THE IRREVERSIBILITY OF RADIATION

An oscillating charge surrounded by an absorbing medium loses energy. Why does radiation have this irreversible character even in a formulation of electrodynamics which is from the beginning symmetrical with respect to the interchange of past and future?

It might at first sight appear that the irreversibility is connected with the property of complete absorption. This is not the case. The expression (37) of the condition of absorption is perfectly symmetrical between advanced and retarded fields. We have only to reverse the roles of these two fields in the derivation following (37) in order to arrive at an equation of motion for the typical particle just as legitimate as (42), and in complete harmony with that equation:

$$m_a \ddot{a}_n = e_a \sum_{k\neq a} F^{(k)}_{n\alpha\,\mathrm{adv}} \dot{a}^\alpha - (2e_a^2/3)(\dot{a}_n \dddot{a}_\alpha - \dddot{a}_n \dot{a}_\alpha)\dot{a}^\alpha. \quad (43)$$

In this equation, however, the force of radiative reaction appears with a sign just opposite to its usual one. Evidently the explanation of the one-sidedness of radiation is not purely a matter of electrodynamics.

We have to conclude with Einstein[11] that the irreversibility of the emission process is a phenomenon of statistical mechanics connected with the asymmetry of the initial conditions with respect to time. In our example the particles of the absorber were either at rest or in random motion before the time at which the impulse was given to the source. It follows that in the equation of motion (42) the sum, $\sum_{k\neq a} F^{(k)}_{n\alpha\,\mathrm{ret}}$, of the retarded fields of the adsorber particles had no particular effect on the acceleration of the source. Consequently the normal term of radiative damping dominates the picture. In the reverse formulation (43) of the equation of motion, the sum of the advanced fields of the absorber particles is not at all negligible, for they are put into motion by the source at just the right time to contribute to the sum $\sum_{k\neq a} F^{(k)}_{n\alpha\,\mathrm{adv}}$. This contribution, apart from the natural random effects of the changes of the absorber, has twice the magnitude of the usual damping term. The negative reactive force of (43) is therefore cancelled out, and a force of the expected sign and magnitude remains.

That it is solely the nature of the initial conditions which governs the direction of the radiation process can be seen by imagining a reversal of the direction of time in the preceding example. We have then a solution of the equations of motion just as consistent as the original solution. However, our interpretation of the solution is different. As the result of chaotic motion going on in the absorber, we see each one of the particles receiving at the proper moment just the right impulse to generate a disturbance which converges upon the source at the precise instant when it is accelerated. The source receives energy and the particles of the absorber are left with diminished velocity. No electrodynamic objection can be raised against this solution of the equations of motion. Small *a priori* probability of the given initial conditions provides our only basis on which to exclude such phenomena.

A comparison of radiation with heat conduction is illuminating. Both processes convert ordered into disordered motion although every elementary interaction involved is microscopically reversible.

Consider for the moment the question of the irreversibility of heat conduction, later to be put into relation with the problem of the one-sidedness of radiation. A portion of matter observed at the present moment to be warmer than its surroundings will cool off in the future with a probability overwhelmingly greater than the chance for it to grow hotter. About the past of the same portion of matter Boltzmann's H-theorem however also predicts an enormously greater likelihood that the body warmed up to its present state rather than cooled down to it. In other words, we are asked to understand the present temperature of the body as the result of a simple statistical fluctuation in the distribution of energy through the entire system. This de-

TABLE I. Decomposition of the symmetric fields of the theory of action at a distance into the fields of the retarded field theory.

Total field acting on ath particle in theory of action at a distance; here decomposed into:	$\sum_{k\neq a}(\frac{1}{2}\overset{(k)}{F}_{\mathrm{ret}}+\frac{1}{2}\overset{(k)}{F}_{\mathrm{adv}})$
(1) Retarded fields of usual formulation of electrodynamics.	$\sum_{k\neq a}\overset{(k)}{F}_{\mathrm{ret}}$
(2) A field completely determined by the motion of the particle itself; denoted as the "radiation field" by Dirac; accounts for the normal force of radiative reaction.	$[\frac{1}{2}\overset{(a)}{F}_{\mathrm{ret}}-\frac{1}{2}\overset{(a)}{F}_{\mathrm{adv}}]_{mn} = (2e_a/3)(\dot{a}_m\dddot{a}_n-\dddot{a}_m\dot{a}_n)$
(3) A residual field, with singularities at none of the particles, but completely determined by the motion of the particles; identified by us with Dirac's "incident field."	$\sum_{\text{all particles}}(\frac{1}{2}\overset{(k)}{F}_{\mathrm{adv}}-\frac{1}{2}\overset{(k)}{F}_{\mathrm{ret}}) \equiv F_{\mathrm{inc}}$

duction is based on the premise that the system was isolated before observation. However, common experience tells us that the given portion of matter probably acquired its abnormal temperature, not via an internal statistical fluctuation, but because it had earlier not been isolated from the outside.

For the radiative analogy of this example of heat conduction, conceive a charged particle bound to a position of equilibrium by a quasi-elastic force. Furthermore suppose its energy at the moment of observation is large in comparison with the agitation of the surrounding absorber particles. There is then an overwhelming probability that the oscillator will lose energy to the absorber at a rate in close accord with the law of radiative damping. What can be said of the particle prior to the moment of acceleration? In an ideal absorbing system completely free of special disturbances, there is an equally overwhelming chance that the energy of the charge was then increasing at a rate given approximately by the inverse of the law of radiative damping. In this case as in heat conduction the abnormally high energy of the object is to be interpreted as the result of a statistical fluctuation. However, that the sun at some past age acquired its energy by such a fluctuation no one now would seriously propose. Obviously the universe is a special system with respect to the origin of which probability considerations cannot freely be applied.

We conclude that radiation and radiative damping come under the head, not of pure electrodynamics, but of statistical mechanics. The conventional expression for the force of radiative reaction, like those for frictional resistance and viscous drag, represents a statistical average only. Application of this concept is not required in such an instance as the case of complete thermodynamical equilibrium, where the relative fluctuations of the actual forces about the conventional values are substantial. The concept of radiative damping is of real value only when we deal with the conversion of organized into disorganized energy, as in wireless transmission or light production.

COMPLETE AND INCOMPLETE ABSORPTION

In the picture of radiation which we have built on the foundation of Tetrode's suggestion, the absorber plays a role of hitherto unsuspected importance. On this account we should investigate not only how much the mechanism depends upon the completeness of the interception, but also the question what should be said of the absorption in the case of the actual universe.

In discussing the case of incomplete interception, we require a convenient means to take into account the initial conditions which so clearly control the irreversibility of the force of radiative reaction. For this reason we shall break down the half-retarded, half-advanced fields of the theory of action at a distance into three parts as shown in Table I. With this decomposition of the field, we arrive at a description of the behavior of a system of particles which is entirely equivalent to the theory of action at a distance but which in the equation of motion,

$$m\ddot{a}_m=e_a\sum_{k\neq a}\overset{(k)}{F}_{m\alpha\,\mathrm{ret}}\dot{a}^\alpha+(2e_a^2/3)\times(\dot{a}_m\dddot{a}_\alpha-\dddot{a}_m\dot{a}_\alpha)\dot{a}^\alpha+e_aF_{m\alpha\,\mathrm{inc}}\dot{a}^\alpha, \quad (44)$$

conceals from view the existence of the advanced part of the fields of Schwarzschild and Fokker. We shall find it convenient to use for the field decomposition of Table I and the dynamical Eq. (44) the term "retarded field formulation of electrodynamics."

The field which enters the third term in the equation of motion (44) vanished in the case of a completely absorbing system. Its appearance in the present case has led us to give it the name of "incident field," which Dirac applied to a quantity having an identical role in the equation of motion. However, on the origin of this field we go beyond Dirac's treatment in giving a prescription for its unique determination in terms of the movements of all the particles of the system. This prescription reveals that the field in question contains the advanced effects of the theory of action at a distance.

Some properties of the incident field may be noted before use is made of this concept in the analysis of special problems. The quantity F_{inc} has a singularity at the site of none of the charged particles. Consequently it satisfies Maxwell's equations for free space. Although completely determined by the motion of the charges, it thus has the character of a disturbance produced by sources at infinity. Now we already have in the retarded field, $\sum_{(\text{all particles})} F_{\text{ret}}^{(k)}$, a quantity whose behavior at all distances is likewise uniquely fixed by the motions of all the charges. Consequently we can expect to be able to deduce the incident field everywhere from a knowledge of the retarded field at large distances from the system of particles. Thus, in the determination of the incident fields we can, if we wish, avoid explicit reference to the movements of the charges, and base our considerations on the asymptotic behavior of their retarded fields alone. This point will be clearer after a consideration of a few examples, and can then be formulated in a general mathematical form as a by-product of an investigation primarily aimed at examining the problem of complete and incomplete absorption.

The simplest example will be the idealized case of a single-charged particle, alone in otherwise charge-free space, which is accelerated either by the gravitational attraction of a passing mass or by some other non-electromagnetic force. For the three electromagnetic forces of the equation of motion (44) we then have the following accounting: (1) There are no other particles, so the retarded field of the first term vanishes. (2) The second term is different from zero and represents the conventional force of radiative reaction. (3) The incident field of the third term is in the present case equal to half the advanced field minus half the advanced field owing to the particle itself. If we imagine the acceleration of the charge to be limited to a short stretch of time, then the incident field represents a disturbance which, long before the moment in question, was converging upon the particle from great distances. It focuses upon the particle at the period of acceleration and subsequently appears as a wave diverging from the charge. This disturbance, apparently produced by sources at infinity, exerts on the particle a force which is just sufficient in magnitude and in sign to cancel the normal force of radiative reaction. The description just given is the rather involved translation into the language of the retarded field theory of the conclusion immediately apparent from the theory of action at a distance with its half-advanced, half-retarded fields; an isolated charge neither experiences a force of radiative reaction nor radiates away electromagnetic energy.

The incident field of the preceding problem could have been determined equally well without knowledge of the motion of the particle itself, by reference to the retarded field, F_{ret}, of the charge at large distances. The latter quantity represents an electromagnetic disturbance which was negligible before the moment of acceleration, and which considerably later than that instant had the character of a diverging spherical wave. We can find a solution, S, of Maxwell's equations for free space, the diverging wave in the asymptotic expansion for which has exactly the same behavior as the field $-\frac{1}{2}F_{\text{ret}}$. By this condition the solution in question is furthermore uniquely determined. On this account it must be identical with another field which also satisfies Maxwell's equations for free space and has for its diverging wave at large distances the same form as $-\frac{1}{2}F_{\text{ret}}$; namely, the incident field, $F_{\text{inc}} \equiv \frac{1}{2}F_{\text{adv}} - \frac{1}{2}F_{\text{ret}}$. Consequently we may write $F_{\text{inc}} = S$, where S is the solution of Maxwell's equations defined as above. From this means of arriving at the value of the incident field we conclude that the incident field is that solution of the wave equation for free space which, when added to the known retarded field, F_{ret}, will reduce by one-half the strength of the diverging wave in the asymptotic representation of F_{ret}.

As next idealized example of incomplete absorption we consider a source at the center of a blackbody with two opposed openings out into

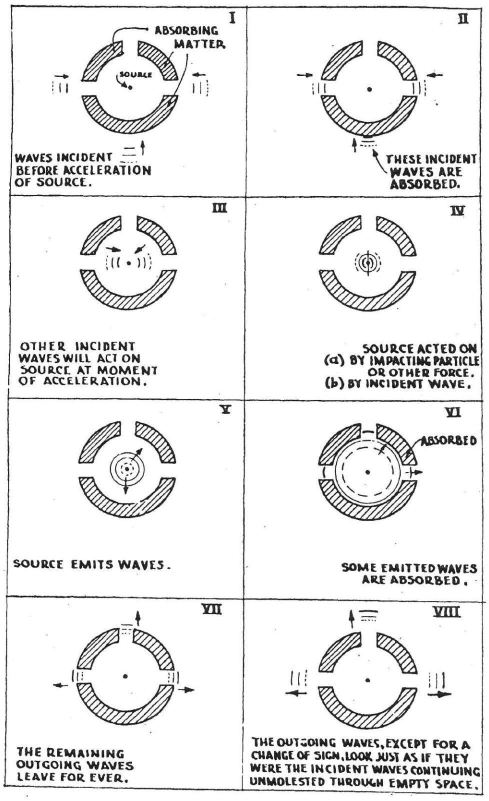

FIG. 1. Advanced effects in two examples of an incompletely absorbing system.

charge-free space (Fig. 1). In those directions not shielded by the absorber an incident field—to use the language of the retarded field theory—will enter, converge on the radiating particle, diverge and go out to infinity, as in the preceding example. This time, however, the wave incident from one side covers but the amount Ω of the whole solid angle of 4π. Consequently at the instant when it is focused upon the source, it reduces the force of radiative reaction only fractionally below the conventional value of this force. The fraction in question is equal in the case of a pair of small opposed openings and a slowly moving source to the product of the following factors:

$\Omega/4\pi$,

the fraction of the whole solid angle spanned by one hole.

2,

factor allowing for the existence of the two openings.

$\frac{3}{2}\sin^2(\mathfrak{A}, r)$,

factor allowing for the orientation of the holes relative to the direction of acceleration. Here $(\mathfrak{A}, r)$ is the angle between the vectorial acceleration of the source and the vector from this particle to one opening. The given expression for the polarization factor assumes that the quantities $d\mathfrak{A}/dt$ and $\mathfrak{A}$ are parallel, and has to be replaced by a more complicated term when this parallelism does not exist.

When the source is moving at the time of acceleration with a speed comparable to that of light, then the angular distribution of the radiation is represented by an expression more complicated than $\sin^2(\mathfrak{A}, r)$ but the general principle is the same.

In the present case of an absorber complete except for two inversion-symmetric openings to charge-free space we conclude that there is a continuous transition, as the size of the apertures is increased, from the full conventional force of radiative reaction on a central source, to the case of no radiative reaction at all. Furthermore, we note that a test charge placed in one of the two openings will receive a disturbance some time before as well as some time after the moment when the source itself is given its acceleration. Generally we may say that the explicit appearance of advanced effects is unavoidable in the case of a system which is an incomplete absorber. However, in neither of the examples so far examined do advanced fields of the source produce explicit advanced effects on any other particle than a test charge: in the first example, because there is no other charged particle; and in the second case, because the incident field is restricted to a region of space where there are no particles to be disturbed, except a possible test particle.

ADVANCED EFFECTS ASSOCIATED WITH INCOMPLETE ABSORPTION

Recognizable advanced effects appear for the first time in a third example, a source at the center of a cavity completely absorbing except for a single passage to charge-free space. (See Fig. 1.) For simplicity we consider the source to be a slowly moving particle. Also we shall denote as "antipassage" that portion of the absorber which is marked out by the inversion, with respect to the source, of the passage itself. Apart from the fields which come from and go to the passage and antipassage, we have the usual solution of the problem of a completely absorbing system. In the language of the retarded formulation of field theory, we can say that there is in all other directions no incident field, and that in those directions the absorber experiences only a normal full strength retarded field as a result of the acceleration of the source. In the language of the theory of action at a distance the result is the same. The source gives out a half-advanced, half-retarded field, the advanced part of which in the direction of the normal portions of the absorber is cancelled by a portion of the advanced fields generated in the absorber itself. The remaining portion of this advanced field combines with the half-strength retarded field of the source to give the full retarded disturbance demanded by experience. The advanced field of the absorber at the location of the source itself produces a force of radiative reaction which is below the conventional amount in proportion only to the small solid angle which we have so far left out of consideration. So far the results are quite as expected.

We have now to consider the effect on the antipassage of what, in the theory of action at a distance, is the half-advanced field of the source. If the passage itself were filled with absorbent, this material would have generated a field, the advanced part of which would have compensated the effect we are now considering. As it is, we visualize two possible solutions of the problem of motion. In the first, the uncompensated half-advanced field of the source sets into motion ahead of time the particles on the inner face of the antipassage. In the second solution, this advanced field is compensated by a mechanism yet to be explained, and the particles in question are not disturbed before the moment of acceleration of the source.

In the first solution, not illustrated in Fig. 1, the particles on the inner face of the antipassage spontaneously accelerate at a moment sufficiently early so that their retarded fields reach the source at just the moment when it is radiating. The retarded field of the charges of the antipassage, evaluated in the cavity, has in the present solution the following properties: (1) It vanishes except in the directions of the passage and antipassage. (2) In those directions it has a value completely determined by the movement of the source. (3) It combines with the Schwarzschild-Fokker field of the source to cancel its outgoing component travelling in the direction of the passage and to build up its advanced component on the side of the antipassage to a full strength advanced wave. The combined advanced wave has a magnitude exactly sufficient to account for the disturbance of particles of the antipassage ahead of time. (4) The field of these particles at the location of the source acts in the opposite sense from the conventional force of radiative reaction. The magnitude of that force is reduced by a fraction which, apart from a polarization factor, is equal to twice the solid angle subtended by the passage, divided by 4π.

The particles of the antipassage, in addition to the anticipatory movements already discussed, undergo, after the moment of acceleration of the source, a disturbance similar to that experienced by the charges which neighbor them on the inner face of the cavity. In this way they are caused to generate fields, the advanced part of which (1) combines with the half-retarded field of the source in the given direction to produce a full retarded disturbance that accounts for the motions in question and (2) cancels the advanced field of the source in the direction of the passage. Thus neither advanced nor retarded disturbance emerges from this passage to be detected by an external test charge.

The self-consistent solution which we have just described in terms of the symmetrical theory of action at a distance is easily summarized in the language of the retarded field theory. The sources of radiation are the central particle and the charges on the inner face of the antipassage. Each is considered to experience the conventional force of radiative reaction and to produce only retarded fields. The fields from the disturbed charges on the inner face of the cavity focus on the central charge at the moment of its acceleration, thus (1) partially compensating the conventional force of radiative reaction and (2) cancelling that part of its retarded field which is travelling in the direction of the passage. No retarded field gets outside the system. The incident field, determinable as we have seen before from the asymptotic behavior of the retarded field, consequently vanishes. Thus in the given illustration the equation of motion of each particle contains only the retarded fields of all the other particles, plus the full conventional force of radiative damping, a conclusion consistent with the solution which we have just given. To complete the picture, we have to express in terms of these equations of motion, the explanation of the early motion of the particles on the inner face of the antipassage. This movement we attribute to the influence of the retarded fields coming from other portions of the wall of the cavity, and reinforcing at just this particular region of the surface. The same type of reasoning can be followed back step by step in the past, along a course which is very like the reversal in time of the mechanism by which a burst of radiative energy dissipates itself. Granted that the existence of incomplete absorption requires, in our example, the explicit occurrence of advanced effects, we have in the given solution one reasonable picture how these advanced effects may build up until the time of disturbance of the source and may then be followed by a succession of retarded effects of magnitudes diminishing as absorption and reflection at the inner walls of the cavity have their effects.

In this third example an equally consistent solution of the problem of advanced effects may be briefly outlined (Fig. 1). The fields are in this case such that a test charge placed within the cavity experiences only the full retarded field of experience. The particles on the wall of the cavity are set into motion only after the time when the source was struck and caused to radiate. These particles, by the now familiar mechanism of absorber response, generate fields, the advanced parts of which in the cavity cancel the advanced field of the source and bring its retarded field up to full strength. In particular, the advanced field of the source in the direction of the passage is compensated, so that an external test charge on that side of the absorber will experience no advanced disturbance. It will, however, on our picture undergo a full strength retarded disturbance. How the half-retarded disturbance of the source in this direction is built up to full strength, and how the half-advanced field in the direction of the antipassage is cancelled, is a question still to be cleared up. For explanation we cannot (1) call upon the advanced fields of the absorber in the direction of the passage, for there is no matter in this direction. Nor can we (2) call upon retarded fields of particles on the inner face of the antipassage for our purpose—although they lie in just the right direction thus to produce the field in question—because they have been assumed not to have been set in motion prior to the time of acceleration of the source. To account for the directional field of so far unknown origin in the cavity, we are by (1) restricted to an explanation in terms of a retarded wave from some source yet to be found, lying in the direction of the antipassage and by (2) this source cannot consist of the particles on the inner face of that portion of the absorber. Consequently we must interpret the field in question as owing to particles set in motion ahead of time on the outer face of the antipassage. This conclusion, like many of the considerations to which we are led in the study of incompletely absorbing systems, appears paradoxical. Nevertheless it leads to a self-consistent solution of our problem. In the first place, the half-retarded field of the surface particles compensates within the antipassage as well as within the cavity the half-advanced field of the source. There is, therefore, no question of the propagation of any disturbance through the thickness of the absorber. Secondly, the half-advanced field of the surface particles appears from the point of view of an external test particle to be a wave front of limited cross section which comes from outer space and which would converge upon the source if the antipassage did not block its path. This half-advanced field in the region exterior to the absorber adds to the half-advanced field from the central source to give a full strength disturbance convergent upon the surface particles in question. Thus an account is given for the force and for the acceleration which they experience ahead of the time of acceleration of the source within the cavity. The energy which is absorbed at the outer surface is later paid out by the source. That accelerated particle experiences the full conventional force of radiative reaction.

The solution just given is readily translated into the language of the retarded formulation of field theory, where the force on each particle is attributed to three sources: the conventional force of damping, the retarded fields of all other particles and an incident field. The incident field has the appearance of a disturbance convergent from outer space upon the external face of the antipassage. By it the particles are accelerated and caused to generate a full strength retarded field which at greater depths within the medium cancels the incident field, a phenomenon which is the normal mechanism of absorption. Thus there is within the cavity no disturbance convergent upon the source, and consequently only the usual retarded effects are observed, part of which also is propagated out through the passage into charge-free space.

In the retarded formulation of field theory, there is no apparent reason for a correlation between the surface absorption and the radiation process within the cavity. The requirement for a connection between the two comes into evidence only in the condition which must be satisfied by the incident field, and which has been discussed above. That there must be such a field follows from the existence of a full strength retarded field, R, diverging outward from the source through the passage in the absorber. Therefore, denote the strength of the incident field in this region of space and at this instant of time by nR, where n is a factor now to be found. Being also divergent from the source, but free of singularities there, the incident field must in this neighborhood and in the given cone of directions be a multiple of the radiation field of the source. On being followed backwards in time to moments previous to the acceleration of the particle, it must, therefore, have in the direction of the antipassage the magnitude $-nA$, where A denotes the advanced solution of Maxwell's equation for the accelerated source. Thus the field incident from great distances upon the particles on the outer surface of the antipassage must have the magnitude $-nA$. These particles generate a retarded field which within the absorber compensates the incident field and therefore has the magnitude $+nA$. This field, followed onward in the direction of the original source, where it naturally has no singularity, at first converges and then diverges to give the appearance of a retarded wave from the source itself. In this neighborhood the retarded field of the surface particles behaves much as does the radiation field of the source. Consequently the strength of the field in question, evaluated in the direction of the passage, is $-nR$. Thus the sum of the retarded fields of all the particles of the system, evaluated outside the passage way, is R (from the source) $-nR$ (from the surface particles). To determine the strength of the incident field, we now apply the condition that the divergent term, nR, in its asymptotic representation must have a strength equal to $-\frac{1}{2}$ times that of the divergent wave owing to the retarded fields of all the particles of the medium:

$$nR = -\tfrac{1}{2}(R - nR). \qquad (45)$$

The solution of this implicit equation gives for the magnitude of the incident wave in the direction of the passage $nR = -R$, and consequently for the strength of the incident wave converging upon the other side of the system $+A$. Thus we check the properties of this second solution of our problem as obtained previously by using the language of half-advanced, half-retarded fields, with no reference to the concept of incident field.

Between the two self-consistent solutions of this third example of an incomplete absorber we make no attempt to choose. We have to accept the fact that the dynamical system in question possesses a number of degrees of freedom which is in direct proportion to the number of particles present. Once it is granted that advanced effects of some kind must be connected with the acceleration of the source, it does not follow uniquely upon which particles these advanced effects must act. The selection is a matter of initial conditions, not of equations of motion. The two solutions so far described are only two relatively simple samples from an infinite number of possible solutions, distinguished from one another by the requirements put upon the initial state of the particles of the absorber. It is only in the case of a completely absorbing system that there is the possibility to find a set of initial conditions which is relatively well determined by statistical considerations.

Self-consistency being the only requirement which has to be met by a solution of the problem of an incomplete absorber, and this requirement in the retarded field formulation being largely contained in a condition to be satisfied by the incident field, it may be of interest to have our so far informal statement of this relation put into mathematical terms. The condition in question furnishes a connection between the incident field, here abbreviated as I, and the sum, R, of the retarded fields of all the particles.

To derive the desired relation, we note that the present formulation of electromagnetic theory expresses the incident field as half the difference between the advanced and retarded fields owing to all the particles. Thus the advanced field of the system is given by the expression $R+2I$. From (1) a knowledge of this advanced field everywhere in space and (2) a knowledge of the values taken on the points, ξ, of a surface surrounding the system by an arbitrary solution, S, of Maxwell's equations for the same charge distribution, we can derive the values of this arbitrary solution at all other points in space, x, from the relation

$$S(x)=(R(x)+2I(x))+(2\pi)^{-1}\left\{\int\int\int d\xi^2 d\xi^3 d\xi^4(\delta\partial S/\partial\xi^1-S\partial\delta/\partial\xi^1)\Big|_{\xi_{-}^{1}}^{\xi_{+}^{1}}+\cdots+\cdots\right.$$
$$\left.-\int\int\int d\xi^1 d\xi^2 d\xi^3(\delta\partial S/\partial\xi^4-S\partial\delta/\partial\xi^4)\Big|_{\xi_{-}^{4}}^{\xi_{+}^{4}}\right\}.^{13} \quad (46)$$

In Eq. (46) the symbol δ stands for the delta function, $\delta(x^\mu-\xi^\mu\cdot x_\mu-\xi_\mu)$. The integral is to be taken only over the immediate neighborhood of those points on the surface from which an advanced wave can reach the point x^1, x^2, x^3 at the time x^4. In the first integrand we have the difference between the values of a certain quantity calculated for the largest and smallest values of ξ^1 consistent with given values of ξ^2, ξ^3, ξ^4; and similarly for the other three integrals. We shall write Eq. (46) symbolically in the form

$$S=(R+2I)+\text{Adv. }[S]. \quad (47)$$

We now apply this general relation to the special half-advanced, half-retarded solution of Maxwell's equations for the system of charges, $S=R+I$. In this way we arrive at an implicit equation by means of which to derive the incident field from the retarded field:

$$0=I+\text{Adv. }[R+I]. \quad (48)$$

This relation is the generalization of Eq. (45) from which we determined the strength of the incident field in the third example above. With this generalization we end our study of the behavior of idealized systems with incomplete absorption and come to the wider question what we should say about the absorbing properties of the system with which we have to deal in nature.

There would be no problem in interpreting the universe as a completely absorbing system if it were an indefinitely extended Euclidean space. The existence of the electron-positron field gives an mechanism by which, even in a vacuum, radiation of some frequencies can undergo absorption processes, and light of all wave-lengths can be scattered. These processes are sufficient ultimately to degrade all the radiation given out by an accelerated charge.

The universe is however now generally regarded as a closed space, in harmony with the illuminating theory put forward by Einstein. In this space present observations suggest that the absorption of radiation is far from complete even at the greatest depths so far plumbed, of the order of one-tenth the calculated radius of the universe. If this conclusion is correct, then a complete electrodynamic description of the mechanism of radiation would require us to take into account not only the curvature of space but also the phenomena summarized under the term "expanding universe." At the present time we know too little about these matters to carry out such a complete description. Moreover, there is yet no compelling reason to attempt this description. We know of course that electrodynamics remains, in other respects as well, to be tied to gravitational phenomena. But we recognize that in this sense our present theory of electrodynamics, like the theories in all other parts of science, is an idealization.

So long as we limit ourselves to the idealization based on the concept of a Euclidean space, we have to consider the question of complete and incomplete absorption on a purely empirical basis. In this connection we will obtain a satis-

[13] The analogue of Eq. (46), for determination of the arbitrary solution from a knowledge of the retarded solution, has been given in a rather different form by W. R. Morgans, Phil. Mag. 9, 148 (1930). The present form is most easily derived by use of the relation

$$(\partial/\partial\xi^\mu)(\partial/\partial\xi_\mu)\delta(x^\nu-\xi^\nu\cdot x_\nu-\xi_\nu)$$
$$=-4\pi\delta(x^1-\xi^1)\delta(x^2-\xi^2)\delta(x^3-\xi^3)\delta(x^4-\xi^4),$$

and application of Green's theorem in four dimensions.

factory account of experience, as we have seen, on the assumption that the universe behaves as a completely absorbing system.

PRE-ACCELERATION

Is there in the case of a completely absorbing universe any consequence of the act of radiation which is so apparently paradoxical as the obviously advanced effects encountered in the instance of an incompletely absorbing system? If so, what words can we reasonably use to assimilate such a phenomenon into our experience?

Any advanced effects to appear in the case of a completely absorbing system must be deducible from the conventional force of radiative reaction, for the only other electrodynamical effects appearing in this case in the equation of motion (42) of the typical particle are the retarded fields of all other particles. That the damping term does lead to an advanced effect follows from an interesting example already considered by Dirac.[9] A source sends a sharp pulse of radiation towards a particle of charge e and mass m. At the instant of arrival the speed of the particle would be expected abruptly to increase if the force of damping were proportional to the first derivative of the displacement. Actually the radiative resistance is proportional to the third derivative of the displacement, and the nature of the solution of the equation of motion is changed. The particle commences to move before the time of arrival of the pulse; and e^2/mc^3 seconds ahead of time it attains a velocity comparable with its final speed.

As a suitable way to speak of this most interesting phenomenon of pre-acceleration brought to light by him, Dirac suggests saying that "it is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure, not of the field equations of electromagnetic theory, but of some of the elementary properties of space-time." This choice of language is perhaps suitable in certain respects to describe the pre-acceleration of the single charge in the example considered by Dirac. It may also be of value in other special instances. However, the given mode of speaking suggests in the case of a medium of closely packed charges the possibility of transmission of signals with a speed greater than that of light over microscopic distances, a conclusion which appears to be denied by a direct investigation of the point. Also the idea that the properties of space time fail in a region of the order of e^2/mc^2 around a charge appear to have possibilities of suggesting misleading conclusions sufficiently great to call for a later search for a more suitable means of expression.

We shall now attempt to test the idea suggested by the term "speed greater than that of light" that the phenomenon of pre-acceleration might be cumulative when charges are spaced at a distance from one another comparable to the quantity e^2/mc^2. The method of analysis will be very nearly that followed by Sommerfeld[14] and Brillouin[15] in their classic resolution of the question how it can be that the speed of propagation of a disturbance in a dispersive medium never exceeds the velocity of light even when the phase velocity for certain frequencies is far above this upper limit. The only significant mathematical difference between the two cases is the change of the damping force from proportionality to the first power of the frequency to proportionality to the third power.

The first step in the procedure of Sommerfeld and Brillouin is to determine the refractive index of the medium, n, as a function of frequency. The charges of the material are assumed normally to be at rest. Consequently the magnetic permeability is unity. According to the standard result of electromagnetic theory the square of the refractive index is in this case equal to the dielectric constant:

$$n^2=\frac{\text{(electric field in a thin slot cut normal to the field)}}{\text{(electric field in a thin cavity cut parallel to the field)}}$$
$$=(E+4\pi P)/E=1+(4\pi P/E). \quad (49)$$

Here P represents the polarization of the medium:

$$P=\text{(number of charges per unit volume)}\cdot\text{(charge of each)}\cdot\text{(displacement from equilibrium)}. \quad (50)$$

[14] A. Sommerfeld, Ann. d. Physik **44**, 177 (1914).
[15] L. Brillouin, Ann. d. Physik **44**, 203 (1914).

For the force which determines the displacement of the charges in a homogeneous isotropic medium it is reasonable, according to Lorentz and Lorenz, to take the result valid for a cavity of spherical form,

$$(\text{force}) = (\text{charge per particle})(E + (4\pi P/3)). \quad (51)$$

The displacement itself is related to the force by the equation of motion,

$$\begin{aligned}(\text{force}) = m(\ddot{\text{displacement}}) \\ +(\text{a constant})\cdot(\text{displacement}) \\ -(2e^2/3c^3)(\dddot{\text{displacement}}). \quad (52)\end{aligned}$$

Here we have visualized in the second term of the right side the possibility that the particles are bound to equilibrium positions by elastic forces. Without such forces we should be led by Earnshaw's theorem to expect that the medium would form a dynamically unstable system. We now follow Lorentz and Lorenz in selecting a function of refractive index which is easy to evaluate as a function of frequency:

$$3(n^2-1)/(n^2+2) = 4\pi P/[E+(4\pi P/3)]$$

$$= \frac{4\pi(\text{particle density})(\text{charge})^2(\text{displacement})}{m(\ddot{\text{disp}}) + (\text{const})(\text{disp}) - (2e^2/3c^3)(\dddot{\text{disp}})}. \quad (53)$$

We consider a monochromatic disturbance of circular frequency $(3mc^3/2e^2)\omega$, and express the elapsed time as $(2e^2/3mc^3)\tau$, where both τ and ω are dimensionless quantities. We assume that the displacement of the typical particle varies with time as $\exp(-i\omega\tau)$. Also we express the number of particles per unit volume in the form $(N/3\pi)(3mc^2/2e^2)^3$, where N is also a magnitude without dimensions. Then from (53) we obtain the refractive index as a function of frequency in the form:

$$n(\omega) = [1+2N/(\omega_0^2-\omega^2-i\omega^3)]^{\frac{1}{2}}. \quad (54)$$

Here we have introduced the abbreviation

$$\omega_0^2 = (2e^2/3mc^3)^2(\text{constant}/m) - (2N/3), \quad (55)$$

where ω_0 is a measure of the extent to which the assumed quasi-elastic force over-compensates the otherwise inherent electrical instability of the system.

The propagation through a vacuum of an electrical disturbance of circular frequency $(3mc^3/2e^2)\omega$ is conveniently described by an electrical field of the form

$$\exp(i\omega\xi - i\omega\tau), \quad (56)$$

when we use the quantity $(2e^2/3mc^2)\xi$ as a measure of distance in the direction of propagation. We suppose this disturbance to be incident on a medium occupying the infinite half-space from $\xi=0$ to $\xi=+\infty$. Then the transverse electric field of the monochromatic wave will be represented in the medium by the expression

$$2(n+1)^{-1}\exp(i\omega n\xi - i\omega\tau). \quad (57)$$

As a measure of the disturbance in the medium we shall take the displacement of the typical particle or, what is up to a constant the same thing, the polarization:

$$\begin{aligned}P &= (n^2-1)E/4\pi, \\ &= (2\pi)^{-1}(n-1)\exp(i\omega n\xi - i\omega\tau). \quad (58)\end{aligned}$$

We are interested in following the progress through the medium, not of a monochromatic wave, but of an initially well defined pulse. We shall idealize the incoming electric field as a delta function, $\delta(\xi-\tau)$, with the property $\delta(u)=0$ when

$$u \neq 0, \quad \int_{-\infty}^{+\infty} \delta(u)du = 1.$$

We recall the representation of the delta function as a superposition of monochromatic waves:

$$\delta(\xi-\tau) = (2\pi)^{-1}\int_{-\infty}^{+\infty} \exp(i\omega\xi - i\omega\tau)d\omega. \quad (59)$$

This expression will represent the electric field in the vacuum. In the medium the electric polarization will accordingly be given as a function of position and time by the integral

$$P(\xi,\tau) = (2\pi)^{-2}\int_{-\infty}^{+\infty}(n(\omega)-1) \times \exp(i\omega n\xi - i\omega\tau)d\omega, \quad (60)$$

where the refractive index is obtained as a function of frequency from (54).

Of the mathematical details of evaluating the polarization of the medium from Eq. (60) it is enough to say that it is convenient to displace the path of integration in the complex plane, and to apply the familiar saddle point method of approximation. This procedure is sufficiently accurate for our purpose when we accept the following reasonable conditions:

(1) We consider a medium of macroscopic dimensions. With the quantity $(2e^2/3mc^2)$ equal to 1.88×10^{-13} cm, it follows that the values of the quantities ξ and τ are of the order of magnitude of 10^{+13}.

(2) The number, N, of particles per volume element $3\pi(2e^2/3mc^2)^3$ is of the order of, or greater than, unity.

(3) At the depth, $\xi(2e^2/3mc^2)$, in the medium, the disturbance, if propagated with the speed of light in vacuum, would arrive at that time, $\tau(2e^2/3mc^3)$, for which $\tau=\xi$; or more briefly, we shall say that the value of the "light-instant," τ, at the depth ξ is given by the equation $\tau=\xi$. We limit our interest to the disturbance at times ahead of the light-instant by an amount which, expressed in the dimensionless measure, $\xi-\tau$, is small in comparison with $\xi\sim 10^{13}$, although this difference may otherwise range all the way from a value very small in comparison with unity to a value as great as several orders of magnitude of 10.

(4) The dimensionless measure of natural frequency of oscillation of the system, ω_0, in order of magnitude is not large in comparison with unity.

Under these conditions we obtain an approximate representation of the polarization of the medium in the form

$$4\pi^2 P \doteqdot (\pi N/\xi)^{\frac{1}{2}}[(1/2N\xi)-(\xi-\tau)+\cdots], \quad (61)$$

for values of $(\xi-\tau)$ in a range of order $1/(N\xi)^{\frac{1}{2}}$ on either side of the light-instant; and

$$4\pi^2 P \doteqdot (4\pi N/3\xi)^{\frac{1}{2}}[(\xi-\tau)/2N\xi]^{\frac{1}{4}} \exp[-(3/4)(2N\xi/\xi-\tau)^{\frac{1}{3}}(\xi-\tau)] \cos[(\pi/3)+(3^{\frac{1}{2}}/4)(2N\xi/\xi-\tau)^{\frac{1}{3}}(\xi-\tau)], \quad (62)$$

for values of $\xi-\tau$ between the rough limits $1/(N\xi)^{\frac{1}{2}}$ on the little side and some small fraction of the quantity $N\xi$ on the big side.

We obtain from expressions (61) and (62) the following picture of the displacement of the charges of the medium at the depth $\xi(2e^2/3mc^2)$ before and at the light-instant:

(1) The typical particle receives a displacement before the light-instant, thus justifying the use of the descriptive term "pre-acceleration" even in the case of a medium containing many particles.

(2) The displacement of the typical particle, instead of increasing with time according to the simple exponential law, $\exp(\tau-\xi)$, derived by Dirac for an isolated particle, is here before the light-instant an oscillatory function of time of much more rapidly increasing amplitude.

(3) The last full oscillation before the light-instant is in the negative sense, that is, opposite to the direction of the field in the original pulse. This oscillation is completed only slightly before the light-instant, so at that time the displacement of the typical particle is positive but small in comparison with its magnitude in the last few preceding vibrations. However, the velocity of the particle at a time about equal to the light-instant has reached the maximum value so far experienced. The condition of approximately zero displacement and high velocity has a certain correspondence with the result which would be expected at the time of arrival of the disturbance in the absence of the phenomenon of pre-acceleration.

(4) The characteristic time of pre-acceleration may reasonably be taken to be measured by the interval between the last two nodes of the oscillation, a quantity which has the order of magnitude $(N\xi)^{-\frac{1}{3}}(2e^2/3mc^2)$, a very small fraction of the so-called classical radius of the charged particle. Another estimate for the time of pri-acceleration of the same order of magnitude is obtained by studying the exponentially increasing envelope of the oscillatory motion described by Eq. (60).

From the tentative conception that the classical radius of a charged particle defines a region within which disturbances are propagated with a speed faster than the velocity of light, it would have appeared reasonable to expect in a very dense medium a macroscopic velocity of propagation significantly greater than the normal limiting value. If this were the case, the interval of pre-acceleration, $\xi-\tau$, would have increased in proportion to the depth, ξ, and would have been appreciable in comparison with ξ. In contrast, we have now found that the characteristic time of pre-acceleration not only decreases slowly with depth in a dense medium, but also is an exceedingly small fraction of the value obtained by Dirac for the case of a single particle. We conclude that it is misleading to attribute the phenomenon of pre-acceleration to an abnormal velocity of light or to a failure of the usual conceptions of space-time in the immediate neighborhood of a charged particle. We are therefore obliged to look to other terms for a suitable way to describe the phenomenon.

PRE-ACCELERATION AS WITNESS TO THE INTERACTION OF PAST AND FUTURE

Pre-acceleration and the force of radiative reaction which calls it forth are both departures from that view of nature for which one once hoped, in which the movement of a particle at a given instant would be completely determined by the motions of all other particles at earlier moments. All thought was excluded of a dependence of the force experienced by the particle upon the future behavior of either that charge itself or any other charges. The past was considered to be completely independent of the future. This idealization is no longer valid when we have a particle commencing to move in anticipation of the retarded fields which have yet to reach it from surrounding charges. Still less is it a good approximation to the truth in the case of an incompletely absorbing system, where we have in addition to the normal damping force an incident field seen above to depend explicitly upon the advanced fields of the individual particles, and where we encounter advanced effects even more striking than preacceleration.

The mechanism by which the future affects the past is illuminated by considering a system of three or more charges in the light of the half-advanced, half-retarded fields of the theory of action at a distance. The retarded field produced by the acceleration of a affects b; the advanced field of b sets c in motion; and c generates a field, the advanced part of which affects a before the moment of its acceleration. By an extension of this line of reasoning it is apparent that the past and future of all particles are tied together by a maze of interconnections. The happenings in neither division of time can be considered to be independent of those in the other. Nevertheless, in a system containing particles sufficient in number to provide effective absorption, an interference takes place between these forces. All the advanced effects are cancelled out except those which are comprised in the conventional force of radiative reaction; and these are limited in their influence to a time of the order of magnitude of the quantity (e^2/mc^3). Therefore, to the extent that the force of radiative reaction can be neglected, we have in the case of a completely absorbing system the possibility to make a cut between past and future; but the cleanness of this cut is limited to times of the order of e^2/mc^3 or greater. Those phenomena which take place in times shorter than this figure require us to recognize the complete interdependence of past and future in nature, an interdependence due to an elementary law of interaction between particles which is perfectly symmetrical between advanced and retarded fields.

SUMMARY

Use of action at a distance with field theory as equivalent and complementary tools for the description of nature has so far been prevented by inability of the first point of view fully to account for the mechanism of radiation. Elucidation of this process in both theories comes from a 23-year old suggestion of Tetrode, that the absorber may be an essential element of the act of emission. A quantitative formulation of this idea is given here on the basis of the following postulates: (1) An accelerated charge in otherwise charge-free space does not radiate energy. (2) The fields which act on a given particle arise only from other particles. (3) These fields are represented by one-half the retarded plus one-half the advanced Lienard-Wiechert solutions of Maxwell's equations.

In a system containing particles sufficient in number ultimately to absorb all radiation, the absorber reacts upon an accelerated charge with a field, the advanced part of which, evaluated in the neighborhood of the source on the basis of these postulates, is found to have the following properties: (1) It is independent of the properties of the absorbing medium. (2) It is completely determined by the motion of the source. (3) It exerts on the source a force which is finite, is simultaneous with the moment of acceleration, and is just sufficient in magnitude and direction to take away from the source the energy which the act of radiation imparts to the surrounding particles. (4) It is equal in magnitude to one-half the retarded field minus one-half the advanced field of the accelerated charge itself, just the field postulated by Dirac as the source of the force of radiative reaction. (5) This field compensates the half-advanced field of the source and combines with its half-retarded field to produce the full retarded disturbance which is required by experience. Radiation is concluded to be a phenomenon as much of statistical mechanics as of pure electrodynamics. A complete correspondence is established between action at a distance and the usual formulation of field theory in the case of a completely absorbing system. In such a system the phenomenon of pre-acceleration appears as the sole evidence of the advanced effects of the theory of action at a distance. Other advanced effects appear in the case of an incompletely absorbing system and are also discussed.